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LETTER TO THE EDITOR

Poisson relations between minors and their consequences

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Abstract. A multiplicative Poisson bracket on $GL(N)$ can be restricted onto $SL(N)$ iff the determinant det is central on $GL(N)$. To decide whether the det is central, a simple criterion is derived in the r-matrix language with the help of a general formula for Poisson brackets between determinants of arbitrary minors.

Quantum group structures on $SL(N)$ always appear through the following device: one starts with a quantum group structure on $GL(N)$, constructs the corresponding quantum determinant, shows that it is central, and then equates this determinant to 1. It is quite likely that all quantum group structures on $SL(N)$ can be obtained in this way, but this is very difficult to prove. In the quasiclassical limit, the corresponding conjecture is that every multiplicative Poisson bracket on $SL(N)$ comes from a multiplicative Poisson bracket on $GL(N)$ with a central determinant. This is undoubtedly true and is, probably, technically feasible to prove by the currently available machinery. (For $N=2$, both the quantum and the quasiclassical claims are proven in (Kupershmidt 1994).) In practice, since the group $GL(N)$ is much more convenient to work with than the group $SL(N)$, one starts with a multiplicative Poisson bracket on $GL(N)$ and then, if the determinant happens to be central, passes on to $SL(N)$ by setting $\det = 1$.

But what is an efficient way to decide whether the det is central? Let $M = (M_i^\alpha)$ be the typical matrix of coordinate functions on $GL(N)$. Let

$$\{M_i^\alpha, M_j^\beta\} = r_{ij}^{\varphi\psi} M_\varphi^\alpha M_\psi^\beta - r_{\varphi\psi}^{\alpha\beta} M_i^\varphi M_j^\psi \tag{1}$$

be the general multiplicative (pre) Poisson bracket on $GL(N)$ (and $\mathrm{Mat}(N)$); summation is always implied over non-fixed repeated indices. We do *not* assume in what follows (unless specified otherwise) that the Jacobi identities for the Poisson bracket (1) are satisfied; thus, the only property required of the structure constants $r_{ij}^{\varphi\psi}$ is the skew-symmetry:

$$r_{ij}^{\varphi\psi} = -r_{ji}^{\psi\varphi}. \tag{2}$$

Let $U = \mathrm{tr}_{\mathrm{L}}(r) = -\mathrm{tr}_{\mathrm{R}}(r)$ be the following matrix in $\mathrm{Mat}(N)$:

$$U_i^j = r_{\varphi i}^{\varphi j} = -r_{i\varphi}^{j\varphi}. \tag{3}$$

The determinantal criterion alluded to above is:

$$\{\det \text{ is central}\} \Leftrightarrow \{U = 0\}. \tag{4}$$

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This follows from the formula

$$\{\det(M), M\} = [U, M]\det(M) \tag{5}$$

meaning, locally, that

$$\{\det(M), M_a^b\} = [U, M]_a^b \det(M). \tag{6}$$

Indeed, from formula (5) we see that det is central iff U is a scalar matrix. But formulae (2), (3) imply that

$$\mathrm{tr}U = 0 \tag{7}$$

and the criterion (4) follows.

Formula (6) results from the following general determinantal identity. For a pair of multi-indices $I = (i_1, i_2, \dots, i_p)$, $J = (j_1, j_2, \dots, j_p)$ with $1 \leqslant i_1, i_2, \dots, j_1, j_2, \dots \leqslant N$, set

$$D_I^J = \det(\mathcal{M}) \qquad \mathcal{M}_\varphi^\psi = M_{i_\varphi}^{j_\psi}. \tag{8}$$

Then

$$\{D_I^J, D_A^B\} = r_{i_k a_l}^{\varphi\psi} D_{I+(\varphi\div i_k)}^J D_{A+(\psi\div a_l)}^B - r_{\varphi\psi}^{j_k b_l} D_I^{J+(\varphi\div j_k)} D_A^{B+(\psi\div b_l)}; \tag{9}$$

here $I + (\varphi \div i_k)$ is the multi-index I whose kth entry i_k is replaced by φ, and similar for the other multi-indices. In particular, when the multi-indices A and B are simple indices: $A = (a)$ and $B = (b)$, formula (9) becomes:

$$\{D_I^J, M_a^b\} = r_{i_k a}^{\varphi\psi} D_{I+(\varphi\div i_k)}^J M_\psi^b - r_{\varphi\psi}^{j_k b} D_I^{J+(\varphi\div j_k)} M_a^\psi. \tag{10}$$

When $I = J = (1, 2, \dots, N)$, formula (10) yields

$$\{\det(M), M_a^b\} = \det(M)(r_{ka}^{k\psi} M_\psi^b - r_{k\psi}^{kb} M_a^\psi) = \det(M)[U, M]_a^b$$

which is formula (6).

Proof of formula (9). Let us first prove formula (10). We use induction on the length $|I|$ of the multi-index I. For $|I| = 1$ formula (10) is just the defining formula (1). To proceed with the induction step, we use Laplace's formula

$$D_{iI}^{\mathcal{J}} = (-1)^{c+1} M_i^{t_c} D_I^{\mathcal{J}-t_c} \tag{11}$$

where iI is the multi-index $(i, i_1, \dots,)$ and $\mathcal{J} - t_c$ is the multi-index $\mathcal{J}$ whose cth entry t_c is removed. Now, for $\mathcal{I} = iI$ we have

$$\{D_{\mathcal{I}}^{\mathcal{J}}, M_a^b\} = \{D_{iI}^{\mathcal{J}}, M_a^b\} = \{(-1)^{c+1} M_i^{t_c} D_I^{\mathcal{J}-t_c}, M_a^b\}[\text{by}(10)]$$

$$= (-1)^{c+1} D_I^{\mathcal{J}-t_c} (r_{ia}^{\varphi\psi} M_\varphi^{t_c} M_\psi^b - r_{\varphi\psi}^{t_c b} M_i^\varphi M_a^\psi) \tag{12.1}$$

$$+ (-1)^{c+1} M_i^{t_c} (r_{i_k a}^{\varphi\psi} D_{I+(\varphi\div i_k)}^{\mathcal{J}-t_c} M_\psi^b - \sum_{s\neq c} r_{\varphi\psi}^{t_s b} D_I^{\mathcal{J}-t_c+(\varphi\div t_s)} M_a^\psi). \tag{12.2}$$

On the other hand, formula (10) predicts that

$$\{D^{\mathcal{J}}_{\mathcal{I}}, M^b_a\} = (r^{\varphi\psi}_{ia} D^{\mathcal{J}}_{\varphi I} + r^{\varphi\psi}_{i_k a} D^{\mathcal{J}}_{iI+(\varphi\div i_k)} M^b_\psi - r^{t_c b}_{\varphi\psi} D^{\mathcal{J}+(\varphi\div t_c)}_{iI} M^\psi_a$$

$$= r^{\varphi\psi}_{ia}(-1)^{c+1} M^{t_c}_\varphi D^{\mathcal{J}-t_c}_I M^b_\psi + r^{\varphi\psi}_{i_k a}(-1)^{c+1} M^{t_c}_i D^{\mathcal{J}-t_c}_{I+(\varphi\div i_k)} M^b_\psi \tag{13.1}$$

$$- r^{t_s b}_{\varphi\psi}\left[\sum_{c\neq s}(-1)^{c+1} M^{t_c}_i D^{\mathcal{J}-t_c+(\varphi\div t_s)}_I + (-1)^{s+1} M^\varphi_i D^{\mathcal{J}-t_s}_I\right] M^\psi_a. \tag{13.2}$$

The 1st, 2nd, 3rd, and 4th terms in the expression (12) match, in the expression (13), the 1st, 4th, 2nd, and 3rd term, respectively. This proves formula (10).

To prove formula (9), we use induction on $|A| = |B|$. The base of the induction is the case $|A| = |B| = 1$, and this is the just proved formula (10). To make an induction step, let $\mathcal{A} = aA$. Then

$$\{D^J_I, D^{\mathcal{B}}_{\mathcal{A}}\} = \{D^J_I, D^{\mathcal{B}}_{aA}\} = \{D^J_I, (-1)^{c+1} M^{\beta_c}_a D^{\mathcal{B}-\beta_c}_A\}[\text{by}(9), (10)]$$

$$= (-1)^{c+1} M^{\beta_c}_a [r^{\varphi\psi}_{i_k a_l} D^J_{I+(\varphi\div i_k)} D^{\mathcal{B}-\beta_c}_{A+(\psi\div a_l)} - \sum_{s\neq c} r^{j_k\beta_s}_{\varphi\psi} D^{J+(\varphi\div j_k)}_I D^{\mathcal{B}-\beta_c+(\psi\div\beta_s)}_A] \tag{14.1}$$

$$+ (-1)^{c+1} D^{\mathcal{B}-\beta_c}_A [r^{\varphi\psi}_{i_k a} D^J_{I+(\varphi\div i_k)} M^{\beta_c}_\psi - r^{i_k\beta_c}_{\varphi\psi} D^{J+(\varphi\div j_k)}_I M^\psi_a]. \tag{14.2}$$

On the other hand, formula (9) predicts that

$$\{D^J_I, D^{\mathcal{B}}_{\mathcal{A}}\} = \{D^J_I, D^{\mathcal{B}}_{aA}\} = r^{\varphi\psi}_{i_k a} D^J_{I+(\varphi\div i_k)} D^{\mathcal{B}}_{\varphi A}$$

$$+ r^{\varphi\psi}_{i_k a_l} D^J_{I+(\varphi\div a_k)} D^{\mathcal{B}}_{aA+(\psi\div a_l)} - r^{j_k\beta_s}_{\varphi\psi} D^{J+(\varphi\div j_k)}_I D^{\mathcal{B}+(\psi\div\beta_s)}_{aA}$$

$$= D^J_{I+(\varphi\div a_k)}[r^{\varphi\psi}_{i_k a}(-1)^{c+1} M^{\beta_c}_\varphi D^{\mathcal{B}-\beta_c}_A + r^{\varphi\psi}_{i_k a_l}(-1)^{c+1} M^{\beta_c}_\varphi D^{\mathcal{B}-\beta_c}_{A+(\psi\div a_l)}] \tag{15.1}$$

$$- r^{j_k\beta_s}_{\varphi\psi} D^{J+(\varphi\div j_k)}_I\left[\sum_{c\neq s}(-1)^{c+1} M^{\beta_c}_a D^{\mathcal{B}-\beta_c+(\psi\div\beta_s)}_A + (-1)^{s+1} M^\psi_a D^{\mathcal{B}-\beta_s}_A\right]. \tag{15.2}$$

The matching scheme is: the 1st, 2nd, 3rd, and 4th term in the expression (14) versus the 2nd, 3rd, 1st and 4th term, respectively, in the expression (15). □

We now consider some applications of the determinantal criterion (4).

Let V be an N-dimensional vector space with a quadratic Poisson bracket

$$\{x_i, x_j\} = c^{kl}_{ij} x_k x_l \qquad c^{kl}_{ij} = -c^{kl}_{ji} = c^{lk}_{ij} \tag{16.1}$$

and suppose $\Lambda^1(V)$ also has a quadratic Poisson bracket

$$\{\xi_i, \xi_j\} = d^{kl}_{ij} \xi_k \xi_l \qquad d^{kl}_{ij} = d^{kl}_{ji} = -d^{lk}_{ij} \tag{16.2}$$

c^{kl}_{ij} and d^{kl}_{ij} are constants, the variables x_is are even, and the variables ξ_is are odd. Then the actions $\text{End}(V) \times V \to V$, $\text{End}(V) \times \Lambda^1(V) \to \Lambda^1(V)$ are Poisson iff formula (1) holds with

$$r^{kl}_{ij} = c^{kl}_{ij} + d^{kl}_{ij}. \tag{16.3}$$

Conversely, given the multiplicative Poisson bracket (1) on End(V), it generates Poisson actions on V and $\Lambda^1(V)$ with

$$c_{ij}^{kl} = (r_{ij}^{kl} + r_{ij}^{lk})/2 \qquad d_{ij}^{kl} = (r_{ij}^{kl} + r_{ji}^{kl})/2. \tag{16.4}$$

In this language, the determinantal criterion (4) becomes

$$c_{\varphi i}^{\varphi j} + d_{\varphi i}^{\varphi j} = 0 \qquad \forall i, j. \tag{17}$$

Let us apply this formula to the Poisson version of the (factorizable) multiparameter deformation of the quantum $GL(N)$ (Sudbery 1990). This is the case

$$\{x_i, x_j\} = \Phi_{ij} x_i x_j \qquad \Phi_{ij} = -\Phi_{ji} \tag{18.1}$$

$$\{\xi_i, \xi_j\} = \Psi_{ij} \xi_i \xi_j \qquad \Psi_{ij} = -\Psi_{ji} \tag{18.2}$$

$$c_{ij}^{kl} = \Phi_{ij}(\delta_{ij}^{kl} + \delta_{ij}^{lk})/2 \qquad d_{ij}^{kl} = \Psi_{ij}(\delta_{ij}^{kl} - \delta_{ij}^{lk})/2 \tag{18.3}$$

$$2r_{ij}^{kl} = \delta_{ij}^{kl}(\Phi_{ij} + \Psi_{ij}) + \delta_{ij}^{lk}(\Phi_{ij} - \Psi_{ij}) \tag{18.4}$$

of formulae (16); here Φ and Ψ are arbitrary skew-symmetric constant matrices. The determinantal criterion (4) in the form (17) then turns into

$$\sum_i (\Phi_{ij} + \Psi_{ij}) = 0 \qquad \forall j. \tag{19}$$

The Jacobi identities for the End(V) with the r-matrix (18.4) are satisfied iff there exists a permutation σ of the indices $(1, 2, \ldots, N)$ such that

$$\Phi_{ij} - \Psi_{ij} = \text{const}[\text{sgn}(\sigma(i) - \sigma(j))]. \tag{20}$$

Let us call the space End(V) *determinantal* if the determinant is central. Set $V^{(1)} =$ End(V).

Theorem 1. $V^{(2)} = \text{End}(V^{(1)})$ is determinantal iff $V^{(1)}$ is.

Proof. The multiplicative Poisson bracket (1) on $V^{(1)} = \text{End}(V)$ induces the following Poisson bracket on $\Lambda^1(V^{(1)}) = \Lambda^1(\text{End}(V))$:

$$\{\Omega_i^\alpha, \Omega_j^\beta\} = r_{ij}^{\varphi\psi}\Omega_\varphi^\alpha \Omega_\psi^\beta - r_{\varphi\psi}^{\alpha\beta}\Omega_i^\varphi \Omega_j^\psi \tag{21}$$

where the Ωs are odd. Formula (21) is equivalent to the property of the pair of actions $\Lambda^1(\text{End}(V)) \times V \to \Lambda^1(V)$ and $\Lambda^1(\text{End}(V)) \times \Lambda^1(V) \to V$ being Poisson. Together, formulae (1) and (21) make into Poisson maps the natural composition

$$\Lambda^u(\text{End}(V)) \times \Lambda^v(\text{End}(V)) \to \Lambda^{u+v}(\text{End}(V)) \qquad u, v \in \{0, 1\} = \mathbf{Z}_2. \tag{22}$$

Formulae (1) and (2) can be put into the form (16), with

$$C\,{}^{\gamma\varepsilon}_{\substack{kl\\ij\\ \alpha\beta}} = (\delta_{\alpha\beta}^{\gamma\varepsilon} r_{ij}^{kl} + \delta_{\beta\alpha}^{\gamma\varepsilon} r_{ij}^{lk} - \delta_{kl}^{ij} r_{\gamma\varepsilon}^{\alpha\beta} - \delta_{kl}^{ji} r_{\varepsilon\gamma}^{\alpha\beta})/2 \tag{23.1}$$

$$D\,{}^{\gamma\varepsilon}_{\substack{kl\\ij\\ \alpha\beta}} = (\delta_{\alpha\beta}^{\gamma\varepsilon} r_{ij}^{kl} - \delta_{\beta\alpha}^{\gamma\varepsilon} r_{ij}^{lk} - \delta_{kl}^{ij} r_{\gamma\varepsilon}^{\alpha\beta} + \delta_{kl}^{ji} r_{\varepsilon\gamma}^{\alpha\beta})/2 \tag{23.2}$$

$$R\,{}^{\gamma\varepsilon}_{\substack{kl\\ij\\ \alpha\beta}} = \delta_{\alpha\beta}^{\gamma\varepsilon} r_{ij}^{kl} - \delta_{ij}^{kl} r_{\gamma\varepsilon}^{\alpha\beta}. \tag{23.3}$$

Hence,

$$U\,{}^{\varepsilon}_{\substack{\gamma\\ \beta\\ j}} = \sum_{i\alpha} R\,{}^{i\varepsilon}_{\substack{\alpha l\\ ij\\ \alpha\beta}} = \dim(V)(\delta_\varepsilon^\beta U_j^l - \delta_j^l U_\varepsilon^\beta). \tag{23.4}$$

In view of the determinantal criterion (4), formula (23.4) proves the claim. $\square$

Remark 1. Formula (6) shows that the determinant det is normalizing. It is very likely that: (1) an arbitrary multiplicative Poisson bracket (1) on the group $GL(N)$ can be deformed into a quantum group structure on $GL(N)$; (2) this structure possesses a multiplicative quantum determinant; and (3) this quantum determinant is normalizing. (All this is true for $N = 2$, it is *not* true for the supergroup $GL(1 \mid 1)$ (Kupershmidt 1993). An infinitesimal counterpart of this is the Drinfel'd 1992) conjecture that every Lie bialgebra can be quantized; this is proved by Reshetikhin (1992).

Remark 2. In general, even when the r-matrix Poisson bracket (1) on End(V) satisfies the Jacobi identity, this is no longer true for the Poisson bracket on End(End(V)) defined by the R-matrix (23.3). In fact, no non-trivial example is known when the Jacobi identities are satisfied for both $V^{(1)}$ and $V^{(2)}$.

Remark 3. Formula (5) is of the Lax type. It implies that

$$\{\det(M), M^k\} = [U, M^k] \det(M)$$

so that

$$\{\det(M), \mathrm{tr}(M^k)\} = 0.$$

This suggests that, more generally,

$$\{\mathrm{tr}(M^l), \mathrm{tr}(M^k)\} = 0 \qquad \forall k, l \in N. \tag{24}$$

This is, indeed, true, and can be deduced from the general formula (9) by the following arguments. Denote by ${}_\lambda D^J_I$ the determinant of the corresponding minor of the matrix ${}_\lambda M = M - \lambda \mathbf{1}$. Then the formula

$$\{\det(M - \lambda \mathbf{1}) \qquad \det(M - \mu \mathbf{1})\} = 0 \tag{25}$$

follows from the following generalization of the formula (9):

$$\begin{aligned}\{{}_\lambda D^J_I, {}_\mu D^B_A\} &= r^{\varphi\psi}_{i_k a_l}\, {}_\lambda D^J_{I+(\varphi\div i_k)}\, {}_\mu D^B_{A+(\psi\div a_l)} - r^{j_k b_l}_{\varphi\psi}\, {}_\lambda D^{J+(\varphi\div j_k)}_I\, {}_\mu D^{B+(\psi\div b_l)}_A + \lambda(-1)^{k+l}\, {}_\lambda D^{J-j_l}_{I-i_k} \\ &\quad \times [r^{j_l\psi}_{i_k a_p}\, {}_\mu D^B_{A+(\psi\div a_p)} - r^{j_l b_q}_{i_k\psi}\, {}_\mu D^{B+(\psi\div b_q)}_A] - \mu(-1)^{p+q}\, {}_\mu D^{B-b_q}_{A-a_p} \\ &\quad \times [r^{b_q\psi}_{a_p i_k}\, {}_\lambda D^J_{I+(\psi\div i_k)} - r^{b_q j_l}_{a_p\psi}\, {}_\lambda D^{J+(\psi\div j_l)}_I].\end{aligned} \tag{26}$$

When $I = J = A = B = (1, 2, \ldots, N)$, each of the 3 summands in formula (26) vanishes, resulting in formula (25).

First, for a matrix $\mathcal{L} \in \mathrm{Mat}(N)$, set

$$(r\mathcal{L})^{\alpha\beta}_{ij} = r^{\mu\beta}_{ij} \mathcal{L}^\alpha_\mu \qquad (\mathcal{L}r)^{\alpha\beta}_{ij} = \mathcal{L}^\mu_i r^{\alpha\beta}_{\mu j}.$$

Then induction on s proves the following formula:

$$\{(M^{s+1})^\alpha_i, M^\beta_j\} = \sum_{p=0}^{s} [(M^p r M^{s+1-p})^{\alpha l}_{ij} M^\beta_l - (M^{s+1-p} r M^p)^{\alpha\beta}_{il} M^l_j].$$

From this we get

$$\{\mathrm{tr}(M^{s+1}), M_j^{\beta}\} = (s+1)[\mathrm{tr_L}(rM^{s+1}), M]_j^l$$

so that

$$\{\mathrm{tr}(M^{s+1}), M\} = (s+1)[\mathrm{tr_L}(rM^{s+1}), M]. \tag{27}$$

This implies, as above, that

$$\{\mathrm{tr}(M^{s+1}), M^l\} = (s+1)[\mathrm{tr_L}(rM^{s+1}), M^l]$$

so that $\{\mathrm{tr}(M^{s+1}), \mathrm{tr}(M^l)\} = 0$, as desired.

Second, for a pair of matrices $\mathcal{L}, \mathcal{N}$ set

$$(\mathcal{L} \otimes \mathcal{N})_{i\alpha}^{j\beta} = \mathcal{L}_i^j \mathcal{N}_\alpha^\beta \qquad \{\mathcal{L} \overset{\otimes}{,} \mathcal{N}\}_{i\alpha}^{j\beta} = \{\mathcal{L}_i^j, \mathcal{N}_\alpha^\beta\}$$
$$[r(\mathcal{L} \otimes \mathcal{L})r]_{ij}^{\alpha\beta} = r_{ij}^{\mu\nu} \mathcal{L}_\mu^\alpha \mathcal{N}_\nu^\beta \qquad [(\mathcal{L} \otimes \mathcal{N})r]_{ij}^{\alpha\beta} = \mathcal{L}_i^\mu \mathcal{N}_j^\nu r_{\mu\nu}^{\alpha\beta}.$$

Then the defining Poisson bracket on $GL(N)$ given by the formula (1) can be written as

$$\{M \overset{\otimes}{,} M\} = [r, M \otimes M] \tag{28}$$

and the induction on s yields

$$\{M^{s+1} \overset{\otimes}{,} M\} = \left[\sum_{p=0}^{s} (M^p \otimes \mathbf{1}) r (M^{s-p} \otimes \mathbf{1}), M \otimes M\right].$$

From this formula the identity (27) results again. More generally still, induction on b shows that

$$\{M^{a+1} \overset{\otimes}{,} M^{b+1}\} = \left[\sum_{p=0}^{a} \sum_{q=0}^{b} (M^p \otimes M^q) r (M^{a-p} \otimes M^{b-q}), M \otimes M\right] \tag{29}$$

and from this identity formula (24) results upon taking the trace of each of the 2 arguments of the tensor product.

Remark 4. Formula (24) is probably known. For the case when the r-matrix $r_{ij}^{\alpha\beta}$ corresponds to the quasiclassical limit of the quantum group $GL_q(n)$, this formula is proved by Ikeda (1991).

Remark 5. For $N > 2$, a power of a quantum $N \times N$ matrix is not, in general, a quantum matrix any more. In addition, the notion of trace does not survive quantization. Thus, formula (24) cannot have a quantum analogue. On the other hand, the notion of determinant survives quantization, so one should expect quantum analogues of the determinantal formulae (9), (25), (26) to exist.

Remark 6. Formulae (9) and (24) are logically independent. This can be seen from the following argument. Taking the differential at **1** of formula (1), we get a Lie algebra structure on $gl(n)^*$, hence a Poisson structure on $C^\infty(gl(n))$ of the form

$$\{N_i^\alpha, N_j^\beta\} = r_{ij}^{\varphi\beta} N_\varphi^\alpha + r_{ij}^{\alpha\varphi} N_\varphi^\beta - r_{\varphi j}^{\alpha\beta} N_i^\varphi - r_{i\varphi}^{\alpha\beta} N_j^\varphi \tag{30.1}$$

or, in tensorial notation,

$$\{N \overset{\otimes}{,} N\} = [r, 1 \otimes N + N \otimes 1]. \tag{30.2}$$

There exists no analogue of the determinantal formula (9) in this framework. However, formula (24) holds true, as follows from taking the traces of the following general formula

$$\begin{aligned}\{N^{p+1} \overset{\otimes}{,} N^{q+1}\} &= \sum_{a+b=p} [(N^a \otimes 1) r (N^b \otimes N^{q+1}) - (N^b \otimes N^{q+1}) r (N^a \otimes 1)] \\ &\quad + \sum_{c+d=q} [(1 \otimes N^c) r (N^{p+1} \otimes N^d) - (N^{p+1} \otimes N^d) r (1 \otimes N^c)].\end{aligned} \tag{31}$$

We see that

$$\{\mathrm{Tr}(N) \overset{\otimes}{,} N\} = [U, N] \tag{32}$$

so that $\det(M)$ is central iff $\mathrm{Tr}(N)$ is.

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